Low dimensional cohomology of general conformal algebras gc_N

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We compute the low dimensional cohomologies $\widetilde{H}^q(gc_N,\mathbb{C})$, $H^q(gc_N,\mathbb{C})$ of the infinite rank general Lie conformal algebras gc_N with trivial coefficients for $q\le 3$, $N=1$ or $q\le 2$, $N\ge 2$. We also prove that the cohomology of gc_N with coefficients in its natural module is trivial, i.e., $H^*(gc_N,\mathbb{C}[\partial]^N)=0$; thus partially solve an open problem of Bakalov-Kac-Voronov in [*Comm. Math. Phys.,* **200** (1999), 561-598].

I. INTRODUCTION

The notion of a conformal algebra, introduced by Kac in Ref. 12, encodes an axiomatic description of the operator product expansion of chiral fields in conformal field theory. Conformal algebras play important roles in quantum field theory and vertex operator algebras (e.g. Ref. 12), whose study has drawn much attention in the literature (e.g. Refs. 1–6, 10, 12–14 and 20–23). As is pointed out in Ref. 2, on one hand, it is an adequate tool for the study of infinite-dimensional Lie algebras satisfying the locality property (cf. Refs. 5, 12 and 14). On the other hand, conformal modules over a conformal algebra R correspond to conformal modules over the associated Lie algebra $\mathrm{Lie}\,R$ (cf. Ref. 3). The main examples of Lie algebras $\mathrm{Lie}\,R$ are the Lie algebras "based" on the punctured complex plane $\mathbb{C}^\times$, namely the Lie algebra $\mathit{Vect}\,\mathbb{C}^\times$ of vector fields on $\mathbb{C}^\times$ (the Virasoro algebra) and the Lie algebra of maps of $\mathbb{C}^\times$ to a finite-dimensional Lie algebra (the loop algebra). Their irreducible conformal modules are the spaces of densities on $\mathbb{C}^\times$ and loop modules, respectively (cf. Ref. 3). Since complete reducibility does not hold in this case (cf. Refs. 4 and 9), one may expect that their cohomology theory is very interesting and important (cf. Ref. 2), just as the cohomology theory of Lie algebras has played important roles in the structure and representation theories of Lie algebras (cf. Refs. 7–9, 11 and 15–19).

A general theory of cohomology of Lie conformal algebras was established by Bakalov, Kac and Voronov in Ref. 2. They also computed the cohomologies for the finite simple Lie conformal algebras. However the problem for the general Lie conformal algebra gc_N, which is an infinite Lie conformal algebra, remains open. It is well-known that the general Lie conformal algebra gc_N plays the same important role in the theory of Lie conformal algebras as the general Lie algebra gl_N does in the theory of Lie algebras: any module $M=\mathbb{C}[\partial]^N$ over a Lie conformal algebra R is obtained via a homomorphism $R\to gc_N$ (cf. Refs. 5 and 12), thus the study of Lie conformal algebras gc_N has drawn some authors' attentions (cf. Refs. 1, 2, 6, 13 and 14). It seems to us that the computation of cohomology of gc_N is important.

In this paper, we compute the low dimensional basic cohomologies $\widetilde{H}^q(gc_N,\mathbb{C})$ and the

reduced cohomology $\widetilde{H}^q(gc_N,\mathbb{C})$ of gc_N with trivial coefficients for $q\leq 3$, $N=1$ or $q\leq 2$, $N\geq 2$. We also prove that the cohomology of gc_N with coefficients in its natural module is trivial, i.e., $H^*(gc_N,\mathbb{C}[\partial]^N)=0$; thus partially solve an open problem in Ref. 2.

In Section II, we shall recall definitions of conformal algebras, their modules and cohomology, and present the main theorem of this paper (Theorem 2.5). Sections III and IV are devoted to the proof of the main theorem.

II. NOTATIONS AND MAIN RESULTS

We shall briefly recall definitions of conformal algebras, their modules and cohomology. More details can be found in, say, Ref. 2.

Definition 2.1: A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module A with a λ-bracket $[a\,{}_\lambda b]$ which defines a linear map $A\times A\to A[\lambda]$, where $A[\lambda]=\mathbb{C}[\lambda]\otimes A$ is the space of polynomials of λ with coefficients in A, satisfying:

$$[\partial a\,{}_\lambda b]=-\lambda[a\,{}_\lambda b],\quad [a\,{}_\lambda \partial b]=(\partial+\lambda)[a\,{}_\lambda b] \qquad \text{(conformal sesquilinearity)}, \tag{2.1}$$

$$[a\,{}_\lambda b]=-[b\,{}_{-\lambda-\partial}a] \qquad \text{(skew-symmetry)}, \tag{2.2}$$

$$[a\,{}_\lambda[b\,{}_\mu c]]=[[a\,{}_\lambda b]_{\lambda+\mu}c]+[b\,{}_\mu[a\,{}_\lambda c]] \qquad \text{(Jacobi identity)}, \tag{2.3}$$

for $a,b,c\in A$. A subset $S\subset A$ is called a *generating set* if S generates A as a $\mathbb{C}[\partial]$-module. If there exists a finite generating set, then A is called *finite*. Otherwise, it is called *infinite*. □

There is a similar notion of associative conformal algebras, which we shall not introduce in this paper. Below we shall only work with Lie conformal algebras, thus we shorten the term "Lie conformal algebra" to "conformal algebra". The simplest nontrivial conformal algebra is the *Virasoro conformal algebra* Vir, which is a rank one free $\mathbb{C}[\partial]$-module generated by a symbol L such that

$$\mathrm{Vir}=\mathbb{C}[\partial]L,\quad [L\,{}_\lambda L]=(\partial+2\lambda)L. \tag{2.4}$$

Note that using (2.1), it suffices to define λ-brackets on a generating set. Let $N\geq 1$ be an integer. The *general conformal algebra gc_N* can be defined (see, e.g., Ref. 14) as an infinite rank free $\mathbb{C}[\partial]$-module with a generating set

$$S_N=\{J_A^n\,|\,n\in\mathbb{Z}_+,\ A\in gl_N\}, \tag{2.5}$$

where gl_N is the space of $N\times N$ matrices (note that the set S_N is not $\mathbb{C}$-linearly independent, for example, $J_{aA}^m=aJ_A^m$ for $a\in\mathbb{C}$), such that the λ-bracket is defined by

$$[J_A^m\,{}_\lambda J_B^n]=\sum_{s=0}^{m}\binom{m}{s}(\lambda+\partial)^s J_{AB}^{m+n-s}-\sum_{s=0}^{n}\binom{n}{s}(-\lambda)^s J_{BA}^{m+n-s}, \tag{2.6}$$

for $m,n\in\mathbb{Z}_+$, $A,B\in gl_N$, where $\binom{m}{s}=\frac{m(m-1)\cdots(m-s+1)}{s!}$ if $s\geq 0$ and $\binom{m}{s}=0$ otherwise, is the binomial coefficient.

Definition 2.2: A *module* over a conformal algebra A is a $\mathbb{C}[\partial]$-module M with a λ-action $a\,{}_\lambda v$ which defines a map $A\times M\to M[[\lambda]]$, where $M[[\lambda]]$ is the set of formal power series of

λ with coefficients in M, such that

$$a_\lambda(b_\mu v) - b_\mu(a_\lambda v) = [a_\lambda b]_{\lambda+\mu} v, \tag{2.7}$$
$$(\partial a)_\lambda v = -\lambda a_\lambda v, \quad a_\lambda(\partial v) = (\partial+\lambda) a_\lambda v, \tag{2.8}$$

for $a, b \in A$, $v \in M$. If $a_\lambda v \in M[\lambda]$ for all $a \in A$, $v \in M$, then the A-module M is called *conformal*. If M is finitely generated over $\mathbb{C}[\partial]$, then M is simply called *finite*.

Below we shall only consider "conformal modules", thus we drop the word "conformal" and simply call a "conformal module" a "module". Clearly, the one-dimensional vector space $\mathbb{C}$ can be defined as a module (called a *trivial module*) over any conformal algebra A with both the action of ∂ and the action of A being zero. Furthermore, for $a \in \mathbb{C}$, $a \neq 0$, one can define a $\mathbb{C}[\partial]$-module $\mathbb{C}_a$, which is the one-dimensional vector space $\mathbb{C}$ such that $\partial v = av$ for $v \in \mathbb{C}_a$. Then $\mathbb{C}_a$ becomes an A-module with trivial action of A.

Let $\alpha \in \mathbb{C}$. The space $\mathbb{C}^N[\partial]$ (a rank N free $\mathbb{C}[\partial]$-module) can be defined as a gc_N-module with λ-action

$$J^m_A{}_\lambda v = (\partial+\lambda+\alpha)^m A v \quad \text{for} \quad A \in gl_N,\ m \in \mathbb{Z}_+,\ v \in \mathbb{C}^N, \tag{2.9}$$

(cf. the statement after (2.4)). We denote this module by $\mathbb{C}^N_\alpha[\partial]$. When $\alpha = 0$, the module $\mathbb{C}^N[\partial] = \mathbb{C}^N_0[\partial]$ is called the *natural module of* gc_N.

Definition 2.3: Let $q \in \mathbb{Z}_+$. A *q-cochain of a conformal algebra A with coefficients in a module M* is a $\mathbb{C}$-linear map $\gamma : A^{\otimes q} \to M[\lambda_1, ..., \lambda_q]$,

$$\gamma(a_1 \otimes \cdots \otimes a_q) = \gamma_{\lambda_1,...,\lambda_q}(a_1, ..., a_q), \tag{2.10}$$

satisfying

$$\gamma_{\lambda_1,...,\lambda_q}(a_1, ..., \partial a_i, ...a_q) = -\lambda_i \gamma_{\lambda_1,...,\lambda_q}(a_1, ..., a_i, ..., a_q) \ \ \text{(conformal antilinearity)}, \tag{2.11}$$
$$\begin{aligned}&\gamma_{\lambda_1,...,\lambda_{i-1},\lambda_{i+1},\lambda_i,\lambda_{i+2},...,\lambda_q}(a_1, ..., a_{i-1}, a_{i+1}, a_i, a_{i+2}, ..., a_q)\\ &\quad = -\gamma_{\lambda_1,...,\lambda_{i-1},\lambda_i,\lambda_{i+1},\lambda_{i+2},...,\lambda_q}(a_1, ..., a_{i-1}, a_i, a_{i+1}, a_{i+2}, ..., a_q) \ \ \text{(skew-symmetry)},\end{aligned} \tag{2.12}$$

for $a_1, ..., a_q \in A$ and all possible i. We let $A^{\otimes 0} = \mathbb{C}$, so that a 0-cochain γ is simply an element of M. □

We define a *differential d of a cochain* γ as follows:

$$\begin{aligned}&(d\gamma)_{\lambda_1,...,\lambda_{q+1}}(a_1, ..., a_{q+1})\\ &= \sum_{i=1}^{q+1} (-1)^{i+1} a_i{}_{\lambda_i} \gamma_{\lambda_1,...,\hat{\lambda}_i,...,\lambda_{q+1}}(a_1, ..., \hat{a}_i, ..., a_{q+1})\\ &\quad + \sum_{1 \le i < j \le q+1} (-1)^{i+j} \gamma_{\lambda_i+\lambda_j,\lambda_1,...,\hat{\lambda}_i,...,\hat{\lambda}_j,...,\lambda_{q+1}}([a_i{}_{\lambda_i} a_j], a_1, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_{q+1}),\end{aligned} \tag{2.13}$$

where, γ is extended linearly over the polynomials in λ_i, and where, the symbol ˆ means the element below it is missing. In particular,

$$(d\gamma)_\lambda(a) = a_\lambda \gamma \quad \text{if} \quad \gamma \in M \quad \text{is a 0-cochain.} \tag{2.14}$$

By Ref. 2, the operator d preserves the space of cochains and $d^2 = 0$, so that the cochains form a complex, which will be denoted by $\widetilde{C}^* = \widetilde{C}^*(A, M) = \oplus_{q\in\mathbb{Z}_+}\widetilde{C}^q(A, M)$, and called the *basic complex* for the A-module M.

Define the structure of a $\mathbb{C}[\partial]$-module on $\widetilde{C}^*(A, M)$ by

$$(\partial\gamma)_{\lambda_1,...,\lambda_q}(a_1, ..., a_q) = (\partial_M + \sum_{i=1}^{q}\lambda_i)\gamma_{\lambda_1,...,\lambda_q}(a_1, ..., a_q), \tag{2.15}$$

where ∂_M denotes the action of ∂ on M. Then $d\partial = \partial d$ (see Ref. 2) and so the graded subspace $\partial\widetilde{C}^* \subset \widetilde{C}^*$ forms a subcomplex. Define the quotient complex $C^* = C^*(A, M) = \widetilde{C}^*(A, M)/\partial\widetilde{C}^*(A, M) = \oplus_{q\in\mathbb{Z}_+}C^q(A, M)$, called the *reduced complex*.

Definition 2.4: The *basic cohomology $\widetilde{H}^*(A, M)$ of a conformal algebra A with coefficients in a module M* is the cohomology of the basic complex $\widetilde{C}^*$. The *(reduced) cohomology* $H^*(A, M)$ is the cohomology of the reduced complex C^*. □

Note that the basic cohomology $\widetilde{H}^*(A, M)$ is naturally a $\mathbb{C}[\partial]$-module, whereas the reduced cohomology $H^*(A, M)$ is a complex vector space.

The main results of this paper is the following theorem.

Theorem 2.5: *(1) For the general conformal algebra gc_1, we have*

$$\dim \widetilde{H}^q(gc_1, \mathbb{C}) = \begin{cases} 1 & \textit{if } \ q = 0 \ \textit{or } 3, \\ 0 & \textit{if } \ q = 1 \ \textit{or } 2, \end{cases} \tag{2.16}$$

and

$$\dim H^q(gc_1, \mathbb{C}) = \begin{cases} 1 & \textit{if } \ q = 0, \ 2 \ \textit{or } 3, \\ 0 & \textit{if } \ q = 1; \end{cases} \tag{2.17}$$

(2) Equations (2.16) and (2.17) also hold for the general conformal algebra gc_N if $q \le 2$;

(3) $H^*(gc_N, \mathbb{C}_a) = 0$ if $a \ne 0$;

(4) $H^*(gc_N, \mathbb{C}^N_\alpha[\partial]) = 0$ for $\alpha \in \mathbb{C}$. Furthermore, for any gc_N-module M which is freely generated over $\mathbb{C}[\partial]$ such that there exists nonzero $c \in \mathbb{C}$ satisfying $J^0_I{}_\lambda v|_{\lambda=0} = cv$ for $v \in M$, where I is the $N \times N$ identity matrix, we have $H^*(gc_N, M) = 0$.

Remark 2.6: (1) Equations (2.16) and (2.17) show that the cohomologies $\widetilde{H}^q(gc_1, \mathbb{C})$, $H^*(gc_1, \mathbb{C})$, $q \le 3$, of the general conformal algebra gc_1 with trivial coefficients are isomorphic to those of the Virasoro conformal algebra with trivial coefficients.

(2) Theorem 2.5(2) in particular shows that there is a unique nontrivial universal central extension of the general conformal algebra gc_N, which agrees with that of the Lie algebra $\mathcal{D}^N$ of $N \times N$ matrix differential operators on the circle (cf. Refs. 16 and 18. It is well-known that $\mathcal{D}^N$ is the distribution Lie algebra associated with gc_N, cf. Ref. 14). A nontrivial reduced 2-cocycle ψ' of gc_N is given in (3.36), and the universal central extension $\widetilde{gc}_N$ of gc_N

corresponding to ψ' is given by

$$\begin{aligned}[][J_A^m\,{}_\lambda J_B^n] = &\sum_{s=0}^{m}\binom{m}{s}(\lambda+\partial)^s J_{AB}^{m+n-s} - \sum_{s=0}^{n}\binom{n}{s}(-\lambda)^s J_{BA}^{m+n-s} \\ &+(-1)^n\frac{m!n!}{(m+n+1)!}\mathrm{tr}(AB)\lambda^{m+n+1}C,\end{aligned} \tag{2.18}$$

where C is a nonzero *central element* of $\widetilde{gc}_N$ (i.e., $[C\,{}_\lambda a] = [a\,{}_\lambda C] = 0$ for all $a \in \widetilde{gc}_N$) such that $\mathbb{C}C$ is a trivial $\mathbb{C}[\partial]$-module.

(3) In Theorem 2.5(4), note that if we define the 0-bracket by $[a\,{}_0 b] = [a\,{}_\lambda b]|_{\lambda=0}$ for $a, b \in gc_N$, and define the 0-action of gc_N on a module M by $a\,{}_0 v = a\,{}_\lambda v|_{\lambda=0}$ for $a \in gc_N$, $v \in M$, then J_I^0 is *central under* 0*-bracket*, i.e., $[J_I^0\,{}_0 a] = [a\,{}_0 J_I^0]$ for $a \in gc_N$, and so the 0-action of J_I^0 on any indecomposable gc_N-module M is a scalar. $\square$

We shall give the proof of Theorem 2.5 in the next two sections.

III. PROOF OF THEOREM 2.5(2)-(4)

We shall keep notations of the previous section. For a q-cochain $\gamma \in \widetilde{C}^q(A, M)$, we call γ a *q-cocycle* if $d\gamma = 0$; a *q-coboundary* or a *trivial q-cocycle* if there is a $(q-1)$-cochain $\phi \in \widetilde{C}^{q-1}(A, M)$ such that $\gamma = d\phi$. Two cochains γ and ψ are *equivalent* if $\gamma - \psi$ is a coboundary. Denote by $\widetilde{D}^q(A, M)$ and by $\widetilde{B}^q(A, M)$ the spaces of q-cocycles and q-coboundaries respectively. Then by Definition 2.4, we have

$$\widetilde{H}^q(A, M) = \widetilde{D}^q(A, M)/\widetilde{B}^q(A, M) = \{\text{equivalent classes of } q\text{-cocycles}\}. \tag{3.1}$$

We shall divide the proof of Theorem 2.5(2)-(4) into several lemmas (although we are unable to give the general result for gc_N in this paper, Lemmas 3.1-4 below may be helpful in determining $\widetilde{H}^*(gc_N, \mathbb{C})$ and $H^*(gc_N, \mathbb{C})$ in the future).

First suppose $\gamma \in \widetilde{C}^q(gc_N, \mathbb{C})$. Clearly, by (2.11), γ is uniquely determined by the right-hand side of (2.10) for $a_1, ..., a_q \in S_N$, where S_N is defined in (2.5). We can regard the right-hand side of (2.10) as a polynomial in $\lambda_1, ..., \lambda_q$. For any fixed $p \in \mathbb{Z}$, we define a $\mathbb{C}$-linear map $\gamma^{(p)} : gc_N^{\otimes q} \to \mathbb{C}[\lambda_1, ..., \lambda_q]$ such that (2.11) holds for $\gamma^{(p)}$ and such that

$$\gamma^{(p)}(J_{A_1}^{n_1} \otimes \cdots \otimes J_{A_q}^{n_q}) = \gamma^{(p)}_{\lambda_1,...,\lambda_q}(J_{A_1}^{n_1}, ..., J_{A_q}^{n_q}), \tag{3.2}$$

is a homogenous polynomial in $\lambda_1, ..., \lambda_q$ consisting of all monomials of total degree p' which appear in $\gamma_{\lambda_1,...,\lambda_q}(J_{A_1}^{n_1}, ..., J_{A_q}^{n_q})$, where

$$p' = p + \sum_{i=1}^{q} n_i. \tag{3.3}$$

Then it is straightforward to see that $\gamma^{(p)} \in \widetilde{C}^q(gc_N, \mathbb{C})$ and

$$\gamma = \sum_{p\in\mathbb{Z}} \gamma^{(p)}. \tag{3.4}$$

Note that (3.4) is possibly an infinite sum, however for given $J^{n_1}_{A_1},...,J^{n_q}_{A_q}\in S_N$, there are only finite many p's such that (3.2) is not zero; we call such a sum *summable.* From (2.6), (2.11) and (2.13) (note that in (2.6), if we informally regard the right-hand side as a polynomial in $\lambda,\partial,J_{AB},J_{BA}$, then it is a homogenous polynomial of the total degree $m+n$; also note that (2.13) now takes the form such that the first sum in the right-hand side is missing since $\mathbb{C}$ is a trivial module and note from (2.11) that when we substitute (2.6) into (2.13), ∂ appeared in (2.6) can be replaced by $-\lambda_i$ for some i), we immediately obtain the following lemma.

Lemma 3.1: A q-cochain $\gamma\in\widetilde{C}^q(gc_N,\mathbb{C})$ is a q-cocycle (resp., q-coboundary) $\Leftrightarrow$ all $\gamma^{(p)}$ are q-cocycles (resp., q-coboundaries). $\square$.

A q-cochain of the form $\gamma^{(p)}$ is called a *homogenous q-cochain of degree p.*

Following Ref. 2, we define an operator $\tau_1:\widetilde{C}^q(gc_N,\mathbb{C})\to\widetilde{C}^{q-1}(gc_N,\mathbb{C})$ as follows: If $q=0$, we set $\tau_1\gamma=0$; otherwise, we set

$$(\tau_1\gamma)_{\lambda_1,...,\lambda_{q-1}}(a_1,...,a_{q-1})=(-1)^{q-1}\tfrac{\partial}{\partial\lambda}\gamma_{\lambda_1,...,\lambda_{q-1},\lambda}(a_1,...,a_{q-1},J)|_{\lambda=0},\tag{3.5}$$

for $a_1,...,a_{q-1}\in S_N$, where $J=J^1_I$ and I is the $N\times N$ identity matrix. Noting that by (2.6),

$$[J^{n_i}_{A_i}\,{}_{\lambda_i}J]=\sum_{s=1}^{n_i}\binom{n_i}{s}(\lambda_i+\partial)^sJ^{n_i+1-s}_{A_i}-(-\lambda_i)J^{n_i}_{A_i},\tag{3.6}$$

we obtain

$$\begin{aligned}&((d\tau_1+\tau_1 d)\gamma^{(p)})_{\lambda_1,...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q})\\ &=(-1)^q\tfrac{\partial}{\partial\lambda}\sum_{i=1}^{q}(-1)^{i+q+1}\gamma^{(p)}_{\lambda_i+\lambda,\lambda_1,...,\hat{\lambda}_i,...,\lambda_q}([J^{n_i}_{A_i}\,{}_{\lambda_i}J],J^{n_1}_{A_1},...,\hat{J}^{n_i}_{A_i},...,J^{n_q}_{A_q})|_{\lambda=0}\\ &=\tfrac{\partial}{\partial\lambda}\sum_{i=1}^{q}\gamma^{(p)}_{\lambda_1,...,\lambda_{i-1},\lambda_i+\lambda,\lambda_{i+1},...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_{i-1}}_{A_{i-1}},[J^{n_i}_{A_i}\,{}_{\lambda_i}J],J^{n_{i+1}}_{A_{i+1}},...,J^{n_q}_{A_q})|_{\lambda=0},\end{aligned}\tag{3.7}$$

where the first equality follows from the fact that all terms appearing in $d\tau_1\gamma^{(p)}$ are cancelled with the corresponding terms in $\tau_1 d\gamma^{(p)}$ and the terms left are all appearing in $\tau_1 d\gamma^{(p)}$ (cf. (2.13)), the second equality follows from (2.12). Note that for a polynomial P, $\frac{\partial P}{\partial\lambda}|_{\lambda=0}$ is simply the coefficient of λ^1 in P. Now we substitute (3.6) into (3.7). By (2.11), $(\lambda_i+\partial)^s$ can be replaced by $(-\lambda)^s$. Since we only need coefficients of λ^1, the terms with $s\geq 2$ in (3.6) do not contribute to the calculation. Thus $[J^{n_i}_{A_i}\,{}_{\lambda_i}J]$ in (3.7) can be replaced by $(\lambda_i-n_i\lambda)J^{n_i}_{A_i}$. Thus (3.7) is equal to

$$\tfrac{\partial}{\partial\lambda}\sum_{i=1}^{q}(\lambda_i-n_i\lambda)\gamma^{(p)}_{\lambda_1,...,\lambda_{i-1},\lambda_i+\lambda,\lambda_{i+1},...\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q})|_{\lambda=0}=p\gamma^{(p)}_{\lambda_1,...\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q}),\tag{3.8}$$

which follows from (3.3) and the fact that for a homogenous polynomial $P(\lambda_1,...,\lambda_q)$ of total degree p', we have

$$\tfrac{\partial}{\partial\lambda}\sum_{i=1}^{q}(\lambda_i-n_i\lambda)P(\lambda_1,...,\lambda_{i-1},\lambda_i+\lambda,\lambda_{i+1},...,\lambda_q)|_{\lambda=0}=(p'-\sum_{i=1}^{q}n_i)P.\tag{3.9}$$

From (3.7) and (3.8), we obtain

$$(d\tau_1+\tau_1 d)\gamma^{(p)}=p\gamma^{(p)}. \tag{3.10}$$

So if $d\gamma=0$, then (3.10) shows that $\gamma'=\sum_{p\neq 0}\gamma^{(p)}=d(\sum_{p\neq 0}p^{-1}\tau_1\gamma^{(p)})$ (note that this is summable, cf. the statement after (3.4)) is a coboundary, and $\gamma-\gamma'=\gamma^{(0)}$. Thus, we obtain the following lemma.

Lemma 3.2: A q-cocycle in $\widetilde{D}^q(gc_N,\mathbb{C})$ is equivalent to a homogenous q-cocycle of degree zero. □

Now suppose γ is a homogenous q-cocycle of degree zero. For $1\leq j,k\leq N$, denote by $E_{j,k}$ the $N\times N$ matrix with entry 1 at (j,k) and 0 otherwise. Then

$$S'_N=\{J^n_{E_{j,k}}\,|\,n\in\mathbb{Z}_+,\ 1\leq j,k\leq N\}, \tag{3.11}$$

is a free generating set of gc_N over $\mathbb{C}[\partial]$. Let $h=\sum_{j=1}^N jE_{j,j}$. We define another operator $\tau_2:\widetilde{C}^q(gc_N,\mathbb{C})\to\widetilde{C}^{q-1}(gc_N,\mathbb{C})$ as follows: We set $\tau_2\gamma=0$ if $q=0$, otherwise we set

$$(\tau_2\gamma)_{\lambda_1,...,\lambda_{q-1}}(a_1,...,a_{q-1})=(-1)^{q-1}\gamma_{\lambda_1,...,\lambda_{q-1},0}(a_1,...,a_{q-1},J^0_h), \tag{3.12}$$

for $a_1,...,a_{q-1}\in S_N$. Now note that by (2.6),

$$[J^{n_i}_{E_{j_i,k_i}}\,{}_{\lambda_i}\,J^0_h]=\sum_{s=0}^{n_i}\binom{n_i}{s}k_i(\lambda_i+\partial)^s J^{n_i-s}_{E_{j_i,k_i}}-j_iJ^{n_i}_{E_{j_i,k_i}}. \tag{3.13}$$

Thus as discussion in (3.7) and (3.8), the terms with $s\geq 1$ do not contribute to the following calculation, and as in (3.7), we have

$$\begin{aligned}&((d\tau_2+\tau_2 d)\gamma)_{\lambda_1,...,\lambda_q}(J^{n_1}_{E_{j_1,k_1}},...,J^{n_q}_{E_{j_q,k_q}})\\ =&\sum_{i=1}^q\gamma_{\lambda_1,...,\lambda_q}(J^{n_1}_{E_{j_1,k_1}},...,J^{n_{i-1}}_{E_{j_{i-1},k_{i-1}}},[J^{n_i}_{E_{j_i,k_i}}\,{}_{\lambda_i}\,J^0_h],J^{n_{i+1}}_{E_{j_{i+1},k_{i+1}}},...,J^{n_q}_{E_{j_q,k_q}})\\ =&\sum_{i=1}^q(k_i-j_i)\gamma_{\lambda_1,...,\lambda_q}(J^{n_1}_{E_{j_1,k_1}},...,J^{n_q}_{E_{j_q,k_q}}).\end{aligned} \tag{3.14}$$

Thus as in Lemma 2.2, we obtain the following lemma.

Lemma 3.3: A q-cocycle in $\widetilde{D}^q(gc_N,\mathbb{C})$ is equivalent to a homogenous q-cocycle γ of degree zero satisfying

$$\gamma_{\lambda_1,...,\lambda_q}(J^{n_1}_{E_{j_1,k_1}},...,J^{n_q}_{E_{j_q,k_q}})=0\quad\textit{if}\quad\sum_{i=1}^q(j_i-k_i)\neq 0. \tag{3.15}$$

□

For a q-cochain $\gamma\in\widetilde{C}^q(gc_N,\mathbb{C})$, we define a linear map $\Delta\gamma: gc_N^{\otimes q}\to\mathbb{C}[\lambda_1,...,\lambda_{q-1}]$ by

$$\Delta\gamma(a_1\otimes\cdots\otimes a_q)=\gamma_{\lambda_1,...,\lambda_q}(a_1,...,a_q)|_{\lambda_q=-\lambda_1-...-\lambda_{q-1}}=\gamma_{\lambda_1,...,\lambda_{q-1},-\lambda_1-...-\lambda_{q-1}}(a_1,...,a_q), \tag{3.16}$$

for $a_1,...,a_q\in gc_N$ (we define $\Delta\gamma=\gamma$ if $q=0$, and define $\Delta\gamma(a_1)=\gamma_{\lambda_1}(a_1)|_{\lambda_1=0}$ if $q=1$). Let $C'^q(gc_N,\mathbb{C})=\{\Delta\gamma\,|\,\gamma\in\widetilde{C}^q(gc_N,\mathbb{C})\}$. Then we obtain a linear map $\Delta:\widetilde{C}^q(gc_N,\mathbb{C})\to$

$C'^q(gc_N,\mathbb{C})$. If $\gamma\in\partial\widetilde{C}^q(gc_N,\mathbb{C})=(\sum_{i=1}^q\lambda_i)\widetilde{C}^q(gc_N,\mathbb{C})$ (note that $\partial_{\mathbb{C}}=0$, cf. (2.15)), then clearly $\Delta\gamma=0$. Thus Δ factors to a map $\Delta:C^q(gc_N,\mathbb{C})\to C'^q(gc_N,\mathbb{C})$.

Lemma 3.4: The map $\Delta:C^q(gc_N,\mathbb{C})\to C'^q(gc_N,\mathbb{C})$ is an isomorphism as spaces.

Proof: Suppose $\Delta\gamma=0$ for a q-cochain γ. For $a_1,...,a_q\in gc_N$, regarding $\gamma_{\lambda_1,...,\lambda_q}(a_1,...,a_q)$ as a polynomial in λ_q, we see that it has a root $\lambda_q=-\sum_{i=1}^{q-1}\lambda_i$, i.e., it is divided by $\sum_{i=1}^q\lambda_i$. Thus

$$\phi(a_1\otimes\cdots\otimes a_q)=(\sum_{i=1}^q\lambda_i)^{-1}\gamma_{\lambda_1,...,\lambda_q}(a_1,...,a_q),\tag{3.17}$$

defines a map $\phi:gc_N^{\otimes N}\to\mathbb{C}[\lambda_1,...,\lambda_q]$. Obviously, ϕ is a q-cochain, and $\gamma=(\sum_{i=1}^q\lambda_i)\phi\in\partial\widetilde{C}^q(gc_N,\mathbb{C})$. $\square$.

Thus we can identify $C^q(gc_N,\mathbb{C})$ with the space $C'^q(gc_N,\mathbb{C})$. We call an element in $C'^q(gc_N,\mathbb{C})$ a *reduced q-cochain*. We define the operator $d:C'^q(gc_N,\mathbb{C})\to C'^{q+1}(gc_N,\mathbb{C})$ by $d\Delta\gamma=\Delta d\gamma$, and then we have similar notions of *reduced q-cocycles, reduced q-coboundaries.*

Lemma 3.5: Theorem 2.5(2) holds.

Proof: Clearly, by (2.14), $\widetilde{D}^0(gc_N,\mathbb{C})=\widetilde{C}^0(gc_N,\mathbb{C})=\mathbb{C}$, and $\widetilde{B}^0(gc_N,\mathbb{C})=0$. Thus $\widetilde{H}^0(gc_N,\mathbb{C})=\mathbb{C}$. Also by (2.15), $\partial\widetilde{C}^0(gc_N,\mathbb{C})=0$ and we have $H^0(gc_N,\mathbb{C})=\mathbb{C}$.

Suppose $\gamma\in\widetilde{C}^1(gc_N,\mathbb{C})$ such that $d\gamma\in\partial\widetilde{C}^2(gc_N,\mathbb{C})$, i.e., there is $\phi\in\widetilde{C}^2(gc_N,\mathbb{C})$ such that

$$\begin{aligned}\gamma_{\lambda_1+\lambda_2}([u_{\lambda_1}v])&=-(d\gamma)_{\lambda_1,\lambda_2}(u,v)=-(\partial\phi)_{\lambda_1,\lambda_2}(u,v)\\&=-(\partial_{\mathbb{C}}+\lambda_1+\lambda_2)\phi_{\lambda_1,\lambda_2}(u,v)=-(\lambda_1+\lambda_2)\phi_{\lambda_1,\lambda_2}(u,v),\end{aligned}\tag{3.18}$$

(cf. (2.13) and (2.15)) for $u,v\in S_N$. By (2.6), we have

$$[J_A^n{}_{\lambda_1}J^0]=\sum_{s=1}^n\binom{n}{s}(\lambda_1+\partial)^sJ_A^{n-s},\tag{3.19}$$

for $A\in gl_N$, $n\in\mathbb{Z}_+$, where $J^0=J_I^0$. Thus by (2.11), (3.18) and (3.19), we have

$$\sum_{s=1}^n\binom{n}{s}(-\lambda_2)^s\gamma_{\lambda_1+\lambda_2}(J_A^{n-s})=\gamma_{\lambda_1+\lambda_2}[J_A^n{}_{\lambda_1}J^0]=-(\lambda_1+\lambda_2)\phi_{\lambda_1,\lambda_2}(u,v).\tag{3.20}$$

Let $\lambda_1=\lambda-\lambda_2$, then expressions in (3.20) are polynomials in λ,λ_2 and the right-hand side is divided by λ, thus each term in the left-hand side is divided by λ. Therefore we can set

$$\gamma'{}_\lambda(J_A^n)=\lambda^{-1}\gamma_\lambda(J_A^n)\quad\text{for}\quad a\in gl_N,\ n\in\mathbb{Z}_+.\tag{3.21}$$

Clearly, (3.21) defines a 1-cochain $\gamma'\in\widetilde{C}^1(gc_N,\mathbb{C})$, and we have $\gamma=\partial\gamma'\in\partial\widetilde{C}^1(gc_N,\mathbb{C})$. This proves that $H^1(gc_N,\mathbb{C})=0$.

Now suppose $\gamma\in\widetilde{D}^1(gc_N,\mathbb{C})$ is a 1-cocycle. This means that $\phi=0$ in (3.18) and (3.20), and so, we obtain $\gamma=0$. Thus $\widetilde{H}^1(gc_N,\mathbb{C})=0$.

Next suppose $\psi \in \widetilde{D}^2(gc_N, \mathbb{C})$ is a homogenous 2-cocycle of degree zero. We define a 1-cochain f which is uniquely determined by

$$f_{\lambda_1}(J_A^n) = (n+1)^{-1} \tfrac{\partial}{\partial \lambda} \psi_{\lambda_1,\lambda}(J_A^{n+1}, J^0)|_{\lambda=0}. \tag{3.22}$$

Set $\gamma = \psi + df$, which is also a homogenous 2-cocycle of degree zero. Then

$$\tfrac{\partial}{\partial \lambda}\gamma_{\lambda_1,\lambda}(J_A^n, J^0)|_{\lambda=0} = \tfrac{\partial}{\partial \lambda}\psi_{\lambda_1,\lambda}(J_A^n, J^0)|_{\lambda=0} - \tfrac{\partial}{\partial \lambda} f_{\lambda_1+\lambda}([J_A^n{}_{\lambda_1} J^0])|_{\lambda=0} = 0, \tag{3.23}$$

where, the last equality follows from (3.19), (2.11) and (3.22) if $n \geq 1$, or from the fact that $\psi_{\lambda_1,\lambda}(J_A^0, J^0)$ is a constant polynomial (cf. (3.3)) if $n = 0$. Thus we have

$$\begin{aligned} 0 =& \tfrac{\partial}{\partial \lambda}(d\gamma)_{\lambda_1,\lambda_2,\lambda}(J_A^m, J_B^n, J^0)|_{\lambda=0} \\ =& \tfrac{\partial}{\partial \lambda}(-\gamma_{\lambda_1+\lambda_2,\lambda}([J_A^m{}_{\lambda_1} J_B^n], J^0) + \gamma_{\lambda_1+\lambda,\lambda_2}([J_A^m{}_{\lambda_1} J^0], J_B^n) - \gamma_{\lambda_2+\lambda,\lambda_1}([J_B^n{}_{\lambda_2} J^0], J_A^m))|_{\lambda=0} \\ =& m\gamma_{\lambda_1,\lambda_2}(J_A^{m-1}, J_B^n) + n\gamma_{\lambda_1,\lambda_2}(J_A^m, J_B^{n-1}), \end{aligned} \tag{3.24}$$

for $A, B \in gl_N$, $m, n \in \mathbb{Z}_+$, where the second equality follows from (2.13), the last equality follows from (3.23), (3.19) and (2.11). Induction on $n \geq 0$ in (3.24) proves $\gamma_{\lambda_1,\lambda_2}(J_A^m, J_B^n) = 0$. Thus $\gamma = 0$ and so $\widetilde{H}^2(gc_N, \mathbb{C}) = 0$.

Finally, suppose $\psi' = \Delta\psi \in C'^2(gc_N, \mathbb{C})$ is a reduced 2-cochain. By (2.13) and (3.16),

$$(d\psi')_{\lambda_1,\lambda_2}(a_1, a_2, a_3) = -\psi'_{\lambda_1+\lambda_2}([a_1{}_{\lambda_1} a_2], a_3) + \psi'_{-\lambda_2}([a_1{}_{\lambda_1} a_3], a_2) - \psi'_{-\lambda_1}([a_2{}_{\lambda_2} a_3], a_1), \tag{3.25}$$

for $a_1, a_2, a_3 \in gc_N$. We define a reduced 1-cochain $f' = \Delta f \in C'^1(gc_N, \mathbb{C})$ as follows (note from (3.16) that $f'(a) = f_\lambda(a)|_{\lambda=0} = f_0(a)$ is simply a linear function $f' : gc_N \to \mathbb{C}$, and it is not necessary to write down explicitly its representative (basic) 1-cochain f)

$$f'(J_A^m) = (m+1)^{-1} \tfrac{d}{d\lambda}\psi'_\lambda(J_A^m, J)|_{\lambda=0}, \tag{3.26}$$

(recall (3.6) that $J = J_I^1$) for $A \in gl_N$, $m \in \mathbb{Z}_+$ (note from (2.11) that $f'(\partial a) = f_0(\partial a) = 0$). By (2.13) and (3.16),

$$(df')_\lambda(a_1, a_2) = -f'([a_1{}_\lambda a_2]), \tag{3.27}$$

for $a_1, a_2 \in gc_N$.

Now suppose ψ' is a reduced 2-cocycle. Then $\gamma' = \psi' + df'$ is a reduced 2-cocycle equivalent to ψ'. By (3.26), (3.27) and (3.6),

$$\tfrac{d}{d\lambda}\gamma'_\lambda(J_A^m, J)|_{\lambda=0} = 0 \text{ for } A \in gl_N,\ m \in \mathbb{Z}_+. \tag{3.28}$$

Thus by (3.25),

$$\begin{aligned} 0 =& \tfrac{\partial}{\partial \lambda}(d\gamma')_{\lambda_1,\lambda}(J_A^m, J_B^n, J)|_{\lambda=-\lambda_1} \\ =& \tfrac{\partial}{\partial \lambda}(-\gamma'_{\lambda_1+\lambda}([J_A^m{}_{\lambda_1} J_B^n], J) + \gamma'_{-\lambda}([J_A^m{}_{\lambda_1} J], J_B^n) - \gamma'_{-\lambda_1}([J_B^n{}_\lambda J], J_A^m))|_{\lambda=-\lambda_1} \\ =& \tfrac{\partial}{\partial \lambda}((m(\lambda_1+\lambda)+\lambda_1)\gamma'_{-\lambda}(J_A^m, J_B^n) - ((n(\lambda+\lambda_1)+\lambda)\gamma'_{-\lambda_1}(J_B^n, J_A^m))|_{\lambda=-\lambda_1}, \end{aligned} \tag{3.29}$$

where the last equality follows from (3.28) and (3.6) (similarly to the discussion after (3.7), $\lambda_1 + \partial$ and $\lambda + \partial$ can be replaced by $\lambda + \lambda_1$ and the terms with $s \geq 2$ do not contribute to the calculation). Using (2.12) and (3.16), the right-hand side of (3.29) is equal to

$$(m+n+1)\gamma'_{\lambda_1}(J_A^m, J_B^n) - \lambda_1 \tfrac{\partial}{\partial\lambda_1}\gamma'_{\lambda_1}(J_A^m, J_B^n) = 0. \tag{3.30}$$

From (3.30), we obtain

$$\gamma'_\lambda(J_A^m, J_B^n) = c_{A,B}^{(m,n)}\lambda^{m+n+1} \quad \text{for some} \quad c_{A,B}^{(m,n)} \in \mathbb{C}. \tag{3.31}$$

In particular,

$$\tfrac{d}{d\lambda}\gamma'_\lambda(J_A^m, J^0)|_{\lambda=0} = \delta_{m,0}c_A, \tag{3.32}$$

where $c_A = c_{A,I}^{(0,0)}$. Similarly to (3.29) (also cf. (3.24)),

$$\begin{aligned}
0 &= \tfrac{\partial}{\partial\lambda}(d\gamma')_{\lambda_1,\lambda}(J_A^m, J_B^n, J^0)|_{\lambda=-\lambda_1} \\
&= -\tfrac{\partial}{\partial\lambda}\gamma'_{\lambda_1+\lambda}([J_A^m\,{}_{\lambda_1} J_B^n], J^0)|_{\lambda=-\lambda_1} + m\gamma'_{\lambda_1}(J_A^{m-1}, J_B^n) + n\gamma'_{\lambda_1}(J_A^m, J_B^{n-1}) \\
&= -\tbinom{m}{m+n}\lambda_1^{m+n}c_{AB} + \tbinom{n}{m+n}(-\lambda_1)^{m+n}c_{BA} + (mc_{A,B}^{(m-1,n)} + nc_{A,B}^{(m,n-1)})\lambda_1^{m+n},
\end{aligned} \tag{3.33}$$

where the last equality follows from (2.6), (2.11), (3.16), (3.31) and (3.32). Taking $m = n = 0$, we obtain $c_{AB} = c_{BA}$. Thus

$$mc_{A,B}^{(m-1,n)} + nc_{A,B}^{(m,n-1)} = (\tbinom{m}{m+n} - (-1)^{m+n}\tbinom{n}{m+n})c_{AB}. \tag{3.34}$$

Thus we solve

$$c_{A,B}^{(m,n)} = (-1)^n \frac{m!n!}{(m+n+1)!}c_{AB} \quad \text{for} \quad A, B \in gl_N,\ m, n \in \mathbb{Z}_+. \tag{3.35}$$

From (3.31) and the fact that $c_A = c_{A,I}^{(0,0)}$ and that $c_{AB} = c_{BA}$, we see that the map $A \mapsto c_A$ is a trace of gl_N, i.e., c_A is a scalar multiple of $\mathrm{tr}(A)$ for $\in gl_N$. Thus (3.31) and (3.35) show that γ' is a multiple of ψ' which is defined by

$$\psi'_\lambda(J_A^m, J_B^n) = (-1)^n \frac{m!n!}{(m+n+1)!}\mathrm{tr}(AB)\lambda^{m+n+1}. \tag{3.36}$$

To see that ψ' is a nontrivial reduced 2-cocycle, first define

$$\psi_{\lambda_1,\lambda_2}(J_A^m, J_B^n) = (-1)^n \frac{m!n!}{(m+n+1)!}((-1)^m\lambda_1^{m+n+1} - (-1)^n\lambda_2^{m+n+1})\mathrm{tr}(AB)\lambda^{m+n+1}. \tag{3.37}$$

Clearly, ψ is a 2-cochain (recall the second sentence in the paragraph before (3.2)), and $\psi' = \Delta\psi$ is a reduced 2-cochain. One can easily check that $d\psi' = 0$ and that $\psi' \neq df'$ for any reduced 1-cochain f'. This proves that $H^2(gc_N, \mathbb{C}) = \mathbb{C}\psi'$. $\square$

Lemma 3.6: Theorem 2.5(3) holds.

Proof: We define an operator $\tau : \widetilde{C}^q(gc_N, \mathbb{C}_a) \to \widetilde{C}^{q-1}(gc_N, \mathbb{C}_a)$ by

$$(\tau\gamma)_{\lambda_1,...,\lambda_{q-1}}(a_1,...,a_{q-1}) = (-1)^{q-1}\gamma_{\lambda_1,...,\lambda_{q-1},\lambda}(a_1,...,a_{q-1},J)|_{\lambda=0}, \tag{3.38}$$

for $a_1,...,a_{q-1} \in gc_N$. Similarly to the discussions in (3.7) and (3.8), we have

$$\begin{aligned}((d\tau+\tau d)\gamma)_{\lambda_1,...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q}) &= (\sum_{i=1}^q \lambda_q)\gamma_{\lambda_1,...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q}) \\ &\equiv -a\gamma_{\lambda_1,...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q}) \ (\mathrm{mod}\, \partial\widetilde{C}^q(gc_N,\mathbb{C}_a)\,),\end{aligned} \tag{3.39}$$

(note that $\partial\widetilde{C}^q(gc_N,\mathbb{C}_a) = (a+\sum_{i=1}^q \lambda_i)\widetilde{C}^q(gc_N,\mathbb{C}_a)$ by (2.15), since $\partial_{\mathbb{C}_a} = a$). Now suppose $\gamma \in \widetilde{C}^q(gc_N,\mathbb{C}_a)$ such that $d\gamma \in \partial\widetilde{C}^{q+1}(gc_N,\mathbb{C}_a)$, i.e., there exists a $(q+1)$-cochain ϕ such that $d\gamma = (a+\sum_{i=1}^{q+1}\lambda_i)\phi$. Clearly, by (3.38) $\tau d\gamma = (a+\sum_{i=1}^q \lambda_i)\tau\phi \in \partial\widetilde{C}^q(gc_N,\mathbb{C}_a)$. Thus (3.39) shows that $\gamma \equiv -d(a^{-1}\tau\gamma)$ $(\mathrm{mod}\,\partial\widetilde{C}^q(gc_N,\mathbb{C}_a)\,)$ is a reduced coboundary (note that we assume $a \neq 0$), i.e., $H^q(gc_N,\mathbb{C}_a) = 0$. □

Lemma 3.7: Theorem 2.5(4) holds.

Proof: Note that as spaces, we have $\mathbb{C}^N_a[\partial][\lambda_1,...,\lambda_q] = \mathbb{C}^N[\lambda_1,...,\lambda_q,\partial]$, and a q-cochain $\widetilde{\gamma} \in \widetilde{C}^q(gc_N,\mathbb{C}^N_\alpha[\partial])$ can be regarded as a map $\widetilde{\gamma} : gc_N^{\otimes q} \to \mathbb{C}^N[\lambda_1,...,\lambda_q,\partial]$,

$$\widetilde{\gamma}(a_1\otimes\cdots\otimes a_q) = \widetilde{\gamma}_{\lambda_1,...,\lambda_q,\partial}(a_1,...,a_q), \tag{3.40}$$

for $a_1,...,a_q \in gc_N$. Regarding (3.40) as a polynomial in $\lambda_1,...,\lambda_q,\partial$ with coefficients in $\mathbb{C}^N$, then similarly to Lemma 3.4, a reduced q-cochain

$$\gamma \in C^q(gc_N,\mathbb{C}^N_\alpha[\partial]) = \widetilde{C}^q(gc_N,\mathbb{C}^N_\alpha[\partial])/(\partial+\sum_{i=1}^q\lambda_i)\widetilde{C}^q(gc_N,\mathbb{C}^N_\alpha[\partial]), \tag{3.41}$$

is uniquely determined by the coefficient of ∂^0 in (3.40). Thus a reduced q-cochain γ can be regarded as a map $\gamma : gc_N^{\otimes q} \to \mathbb{C}^N[\lambda_1,...,\lambda_q]$,

$$\gamma(a_1\otimes\cdots\otimes a_q) = \gamma_{\lambda_1,...,\lambda_q}(a_1,...,a_q). \tag{3.42}$$

Define an operator $\tau_0 : C^q(gc_N,\mathbb{C}^N_\alpha[\partial]) \to C^{q-1}(gc_N,\mathbb{C}^N_\alpha[\partial])$ by (cf. (3.38))

$$(\tau_0\gamma)_{\lambda_1,...,\lambda_{q-1}}(a_1,...,a_{q-1}) = (-1)^{q-1}\gamma_{\lambda_1,...,\lambda_{q-1},\lambda}(a_1,...,a_{q-1},J^0)|_{\lambda=0}. \tag{3.43}$$

Similarly to the discussions in (3.7) and (3.8), using (3.19), we have (comparing with (3.7), all terms corresponding to the right-hand side of (3.7) are now zero because J has been replaced by J^0 and we do not take partial derivative $\frac{\partial}{\partial\lambda}$; but note that since the first sum in (2.13) is not zero in this case, we have one more term here)

$$((d\tau_0+\tau_0 d)\gamma)_{\lambda_1,...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q}) = J^0{}_\lambda\gamma_{\lambda_1,...,\lambda_q}(J^{n_1}_{A_1},...,J^{n_q}_{A_q})|_{\lambda=0}. \tag{3.44}$$

Now by (2.9), the λ-action of gc_N on its module $\mathbb{C}_\alpha^N[\partial]$ in particular satisfies $J^0{}_\lambda v = v$ for $v \in \mathbb{C}_\alpha^N[\partial]$. Thus the right-hand side of (3.44) is simply $-\gamma_{\lambda_1,...,\lambda_q}(J_{A_1}^{n_1}, ..., J_{A_q}^{n_q})$, i.e., we obtain

$$\gamma = (d\tau_0 + \tau_0 d)\gamma. \tag{3.45}$$

In particular, if γ is a reduced cocycle, (3.45) gives that $\gamma = d(\tau_0\gamma)$ is a coboundary, i.e., $H^q(gc_N, \mathbb{C}_\alpha^N[\partial]) = 0$.

Clearly, the above proof works for any gc_N-module M satisfying the condition stated in Theorem 2.5(4). □

Thus Theorem 2.5(2)-(4) is proved.

IV. PROOF OF THEOREM 2.5(1)

This section is devoted to the proof of Theorem 2.5(1). By Lemma 3.5, it remains to consider the case $q = 3$. Since some of the following arguments also work for general q-cocycles, we shall first consider q-cocycles with $q \geq 3$ so that it may be possible to use these arguments to determine higher dimensional cohomologies in the future.

Let $gc = gc_1$. It has a free generating set $S = \{J^n \,|\, n \in \mathbb{Z}_+\}$, such that

$$[J^m{}_\lambda J^n] = \sum_{s=1}^{m} \binom{m}{s} (\lambda+\partial)^s J^{m+n-s} - \sum_{s=1}^{n} \binom{n}{s} (-\lambda)^s J^{m+n-s}, \tag{4.1}$$

for $m, n \in \mathbb{Z}_+$. We shall give some more notations. An element in $\mathbb{Z}_+^q$ is denoted by

$$\underline{n} = \underline{n}[q] = (n_1, ..., n_q), \quad n_1, ..., n_q \in \mathbb{Z}_+, \tag{4.2}$$

(when there is no confusion we denote it by $\underline{n}$, otherwise we denote it by $\underline{n}[q]$). Denote $J^{\underline{n}} = J^{n_1} \otimes \cdots \otimes J^{n_q} = (J^{n_1}, ..., J^{n_q}) \in gc^{\otimes q}$. Denote $\underline{\lambda} = \underline{\lambda}[q] = (\lambda_1, ..., \lambda_q)$. For $\underline{n} \in \mathbb{Z}_+^q$, let $|\underline{n}| = \sum_{i=1}^q n_i$, called the *level of* $\underline{n}$. We define a total ordering on $\mathbb{Z}_+^q$ by the *level-lexicographical order*, i.e.,

$$\underline{m} < \underline{n} \;\Leftrightarrow\; |\underline{m}| < |\underline{n}|, \text{ or } |\underline{m}| = |\underline{n}| \text{ and } \exists\, p \text{ such that } m_i = n_i \text{ for } i < p \text{ and } m_p < n_p, \tag{4.3}$$

for $\underline{m}, \underline{n} \in \mathbb{Z}_+^q$. Set

$$\mathcal{N}_q = \{\underline{n} \in \mathbb{Z}_+^q \,|\, n_1 \leq n_2 \leq ... \leq n_q\}. \tag{4.4}$$

For $m, n \in \mathbb{Z}$, we denote $[m, n] = \{m, m+1, ..., n\}$. Let $\mathcal{S}_q$ be the permutation group on the index set $[1, q]$, which acts on $\mathbb{C}^q$ by $\sigma(v) = (v_{\sigma(1)}, ..., v_{\sigma(q)})$ for $v = (v_1, ..., v_q) \in \mathbb{C}^q$. Then for any $\underline{n} \in \mathbb{Z}_+^q$, there is a unique $\underline{n}^* \in \mathcal{N}_q$ and some $\sigma \in \mathcal{S}_q$ such that $\underline{n}^* = \sigma(\underline{n}) \in \mathcal{N}_q$ and $\underline{n}^* \leq \sigma(\underline{n})$. In fact

$$\underline{n}^* = \min\{\sigma(\underline{n}) \,|\, \sigma \in \mathcal{S}_q\}, \tag{4.5}$$

is the minimal element in $\mathcal{S}_q(\underline{n}) = \{\sigma(\underline{n}) \,|\, \sigma \in \mathcal{S}_q\}$.

A q-cochain γ is uniquely determined by $\gamma_{\underline{\lambda}}(J^{\underline{n}})$ for $\underline{n} \in \mathcal{N}_q$ and

$$\gamma_{\underline{\lambda}}(J^{\underline{n}}) = \mathrm{sgn}(\sigma)\gamma_{\sigma(\underline{\lambda})}(J^{\sigma(\underline{n})}), \tag{4.6}$$

for $\underline{n} \in \mathbb{Z}_+^q$, $\sigma \in \mathcal{S}_q$, where $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ. In fact, $\gamma_{\underline{\lambda}}(J^{\underline{n}})$ can be arbitrary polynomial in $\underline{\lambda}$ satisfying (4.6) for all σ such that $\sigma(\underline{n}) = \underline{n}$.

First we construct a 3-cochain $\bar{\gamma}$ as follows:

$$\bar{\gamma}_{\underline{\lambda}}(J^{\underline{n}}) = \begin{cases} \lambda_2^{n_3} - \lambda_1^{n_3} & \text{if } \ n_1 = n_2 = 0, n_3 \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{4.7}$$

for $\underline{n} \in \mathcal{N}_3$ (note that in the first case, we let $n_3 \neq 0$ in order to avoid the problem on how to deal with 0^0 when we set $\lambda_1 = 0$).

Lemma 4.1: $\bar{\gamma}$ is a nontrivial 3*-cocycle.*

Proof: One can define a *Leibniz q-cochain* by removing the skew-symmetric condition (2.12), and define the *Leibniz differential operator d_L* by changing (2.13) into

$$\begin{aligned} &(d_L\gamma)_{\lambda_1,...,\lambda_{q+1}}(a_1, ..., a_{q+1}) \\ =&\sum_{i=1}^{q+1}(-1)^{i+1}a_{i\,\lambda_i}\gamma_{\lambda_1,...,\hat{\lambda}_i,...,\lambda_{q+1}}(a_1, ..., \hat{a}_i, ..., a_{q+1}) \\ &+\sum_{1\le i<j\le q+1}(-1)^i\gamma_{\lambda_1,...,\hat{\lambda}_i,...,\lambda_{j-1},\lambda_i+\lambda_j,\lambda_{j+1},...,\lambda_{q+1}}(a_1, ..., \hat{a}_i, ..., a_{j-1}, [a_{i\,\lambda_i}a_j], a_{j+1}, ..., a_{q+1}), \end{aligned} \tag{4.8}$$

(note that if γ is a (regular) q-cochain, then (4.8) coincides with (2.13), i.e., $d = d_L$ in this case). Then we obtain *Leibniz cohomology* (cf. Ref. 2). We shall not discuss Leibniz cohomology here, but we define a Leibniz 2-cochain f by

$$f_{\lambda_1,\lambda_2}(J^0, J^0) = 1 \ \text{ and } \ f_{\lambda_1,\lambda_2}(J^m, J^n) = 0 \ \text{ if } \ (m, n) \neq (0, 0). \tag{4.9}$$

One can immediately check that $\bar{\gamma} = d_L f$ (thus $\bar{\gamma}$ is a Leibniz 3-coboundary). Therefore $d\bar{\gamma} = dd_L f = d_L^2 f = 0$, i.e., $\bar{\gamma}$ is a (regular) 3-cocycle. However there is no 2-cochain ϕ such that $d\phi = \bar{\gamma}$ because if $d\phi = \bar{\gamma}$ then we also have $\phi_{\lambda_1,\lambda_2}(J^0, J^0) = 1$ and so ϕ is not a (regular) 2-cochain ((2.12) is not satisfied). Thus $\bar{\gamma}$ is a nontrivial 3-cocycle. □

Now let γ be a q-cocycle with $q \geq 3$. By Lemma 3.2, we can suppose γ is homogenous with degree zero. First we have the following lemma.

Lemma 4.2: If $q = 3$, by replacing γ by $\gamma - c\bar{\gamma}$ for some $c \in \mathbb{C}$, we can suppose $\gamma_{\underline{\lambda}}(J^0, J^0, J) = 0$.

Proof: Note that $\gamma_{\underline{\lambda}}(J^0, J^0, J)$ is a linear polynomial in $\underline{\lambda}$ (cf. (3.3)) which is skew-symmetric with respect to λ_1, λ_2 by (2.12). Thus $\gamma_{\underline{\lambda}}(J^0, J^0, J) = c(\lambda_2 - \lambda_1)$ for some $c \in \mathbb{C}$. Replacing γ by $\gamma - c\bar{\gamma}$, we have the lemma. □

To prove (2.16), our strategy is the following: We want to prove by induction on $\underline{n} \in \mathcal{N}_q$ (with respect to the order (4.3)) that after a number of steps in each of which γ is replaced

by $\gamma-\gamma'$ for some q-coboundaries γ' we obtain that $\gamma_{\underline{\lambda}}(J^{\underline{m}})=0$ for all $\underline{m}\in\mathcal{N}_q$, $\underline{m}\le\underline{n}$ (thus we obtain that $\gamma_{\underline{\lambda}}(J^{\underline{n}})=0$ for all $\underline{n}\in\mathcal{N}_q$, i.e., $\gamma=0$, after a countably infinite number of steps; this amounts to saying that γ is subtracted by an infinite sum of q-coboundaries, but from the following proof we see that this infinite sum is summable, cf. the statement after (3.4)). For the case $q=3$, this will be done by a number of lemmas (unfortunately, not all arguments work for $q\ge4$, cf. the proof of Lemma 4.6).

Lemma 4.3: $\gamma_{\underline{\lambda}}(J^{\underline{n}})=0$ *if* $|\underline{n}|\le1$.

Proof: Note that $\gamma_{\underline{\lambda}}(J^{\underline{n}})$ is a polynomial in $\underline{\lambda}$ on degree $|\underline{n}|$. If $|\underline{n}|=0$, we have $\gamma_{\underline{\lambda}}(J^{\underline{n}})=0$ by (4.6). If $|\underline{n}|=1$, then $\underline{n}=(0,...,0,1)$ and $\gamma_{\underline{\lambda}}(J^{\underline{n}})$ is skew-symmetric with respect to $\lambda_1,...,\lambda_{q-1}$, thus divided by $\prod_{1\le i<j\le q-1}(\lambda_i-\lambda_j)$, which has degree $(q-1)(q-2)/2>1$ if $q>3$. Thus $\gamma_{\underline{\lambda}}(J^{\underline{n}})=0$ if $q>3$. If $q=3$, then $\gamma_{\underline{\lambda}}(J^{\underline{n}})=\gamma_{\underline{\lambda}}(J^0,J^0,J)=0$ by Lemma 4.2. $\square$

Now suppose $|\underline{n}|\ge2$. We set $i_0=\#\{i\in[1,q]\,|\,n_i=0\}\ge0$ (where $\#X$ stands for the size of the finite set X), $i_2=q-\#\{i\in[1,q]\,|\,n_i=n_q\}\le q-1$. If $i_0\ne i_2$, we set i_1 to satisfy

$$0=n_1=...=n_{i_0}<n_{i_0+1}\le...\le n_{i_1}<n_{i_1+1}=...=n_{i_2}<n_{i_2+1}=...=n_q; \tag{4.10}$$

if $i_0=i_2$, we set $i_1=0$.

Let $\underline{m}\in\mathcal{N}_{q+1}$ be such that $|\underline{m}|=|\underline{n}|+1$. Consider $(d\gamma)_{\underline{\lambda}[q+1]}(J^{\underline{m}})$ (cf. notation (4.2)). Note that when we substitute (4.1) into (2.13), using (2.11) and (2.12), we obtain that $(d\gamma)_{\underline{\lambda}[q+1]}(J^{\underline{m}})$ is a combination of $\gamma_{\underline{\lambda}'}(J^{\underline{k}})$ with coefficients being polynomials in $\underline{\lambda}[q+1]$, where $\underline{k}\in\mathcal{N}_q$, $|\underline{k}|\le|\underline{n}|$, and $\underline{\lambda}'=(\lambda'_1,...,\lambda'_q)$ such that each λ'_i is a linear polynomial in $\underline{\lambda}[q+1]$. Using the inductive assumption, $\gamma_{\underline{\lambda}'}(J^{\underline{k}})=0$ if $|\underline{k}|<|\underline{n}|$. Thus the terms with $s\ge2$ in (4.1) do not contribute to (2.13) (cf. the discussion after (3.7)), and so we have (here we use (4.8) instead of (2.13))

$$\begin{aligned}0&=(d\gamma)_{\underline{\lambda}[q+1]}(J^{\underline{m}})\\&=\sum_{1\le i<j\le q+1}(-1)^i(m_j\lambda_i-m_i\lambda_j)\;\gamma_{\lambda_1,...,\hat{\lambda}_i,...,\lambda_{j-1},\lambda_i+\lambda_j,\lambda_{j+1},...,\lambda_{q+1}}(J^{\underline{m}(i,j)}),\end{aligned}\tag{4.11}$$

where

$$\underline{m}(i,j)=(m_1,...,\hat{m}_i,...,m_{j-1},m_i+m_j-1,m_{j+1},...,m_{q+1}), \tag{4.12}$$

and the right-hand side of (4.11) is a combination of $\gamma_{\underline{\lambda}'}(J^{\underline{m}(i,j)^*})$ (cf. (4.5) and (4.6)).

Lemma 4.4: $\gamma_{\underline{\lambda}}(J^{\underline{n}})=0$ *if* $n_1\ge1$ *(i.e.,* $i_0=0$*).*

Proof: In (4.11), take $\underline{m}=(0,n_1,...,n_{q-1},n_q+1)\in\mathcal{N}_{q+1}$. In (4.12), if $i\ne1$, then $m_1=0<n_1$ and so $\underline{m}(i,j)^*\le\underline{m}(i,j)<\underline{n}$; by induction, $\gamma_{\underline{\lambda}'}(J^{\underline{m}(i,j)^*})=0$. Similarly, $\gamma_{\underline{\lambda}'}(J^{\underline{m}(i,j)^*})=0$ if $j\ne q+1$. Thus the only possible nonzero term in (4.11) is the one with $(i,j)=(1,q+1)$. Since $\underline{m}(1,q+1)=\underline{n}$, (4.11) gives

$$-(n_q+1)\lambda_1\gamma_{\lambda_2,...,\lambda_q,\lambda_1+\lambda_{q+1}}(J^{\underline{n}})=0. \tag{4.13}$$

This gives the lemma. $\square$

From now on, we assume that $n_1 = 0$.

Lemma 4.5: $\gamma_{\underline{\lambda}}(J^{\underline{n}}) = 0$ *if* $i_2 = q-1$ *and* $n_q \geq n_{q-1} + 2$ *(cf. (4.10)).*

Proof: As above, now (4.11) gives (cf. (4.13))

$$\sum_{i=1}^{i_0+1}(-1)^i(n_q+1)\lambda_i\gamma_{\lambda_1,...,\hat{\lambda}_i,...,\lambda_q,\lambda_i+\lambda_{q+1}}(J^{\underline{n}}) = 0. \tag{4.14}$$

Replacing $(\lambda_1, ..., \lambda_{q+1})$ by $(\lambda, \lambda_1, ..., \lambda_q)$ and applying the operator $\frac{\partial}{\partial\lambda}|_{\lambda=0}$ to (4.14), we obtain

$$\gamma_{\underline{\lambda}}(J^{\underline{n}}) = -\sum_{i=1}^{i_0}(-1)^i\lambda_i\tfrac{\partial}{\partial\lambda}\gamma_{\lambda,\lambda_1,...,\hat{\lambda}_i,...,\lambda_{q-1},\lambda_i+\lambda_q}(J^{\underline{n}})|_{\lambda=0}. \tag{4.15}$$

We define a $(q-1)$-cochain f as follows

$$f_{\underline{\lambda}[q-1]}(J^{\underline{k}}) = \begin{cases} -n_q^{-1}\frac{\partial}{\partial\lambda}\gamma_{\lambda,\lambda_1,...,\lambda_{q-1}}(J^{\underline{n}})|_{\lambda=0} & \text{if} \quad \underline{k} = \underline{n}^-, \\ 0 & \text{otherwise}, \end{cases} \tag{4.16}$$

for $\underline{k}\in\mathcal{N}_{q-1}$, where $\underline{n}^- = (n_2, ..., n_{q-1}, n_q - 1)\in\mathcal{N}_{q-1}$ (cf. (4.10)). Indeed, f is a $(q-1)$-cochain (cf. the statement after (4.6)): Write $\underline{n}^-$ as $\underline{n}^- = (n_1^-, ..., n_{q-1}^-)$, then $n_i^- = n_j^- \;\Leftrightarrow\; n_{i+1} = n_{i+1}$. Thus the skew-symmetric condition (4.6) for f follows from the skew-symmetric condition for γ. We claim that

$$\gamma_{\underline{\lambda}}(J^{\underline{k}}) = (df)_{\underline{\lambda}}(J^{\underline{k}}), \tag{4.17}$$

for all $\underline{k} \in \mathcal{N}_q$ with $\underline{k} \leq \underline{n}$. If $\underline{k} = \underline{n}$, similarly to (4.14), we have

$$(df)_{\underline{\lambda}}(J^{\underline{n}}) = \sum_{i=1}^{i_0}(-1)^i n_q\lambda_i f_{\lambda_1,...,\hat{\lambda}_i,...,\lambda_{q-1},\lambda_i+\lambda_q}(J^{\underline{n}^-}) = \gamma_{\underline{\lambda}}(J^{\underline{n}}), \tag{4.18}$$

where the last equality follows from (4.15) and (4.16). If $\underline{k} < \underline{n}$, when we substitute (4.1) into (2.13) for $(df)_{\underline{\lambda}}(J^{\underline{k}})$, as in (4.11), $(df)_{\underline{\lambda}}(J^{\underline{k}})$ is a combination of the form $f_{\underline{\lambda}'}(J^{\underline{k}(i,j)})$, and we see that $\underline{k}(i,j) < \underline{n}^-$ (cf. (4.12)), i.e., the term $f_{\underline{\lambda}'}(J^{\underline{n}^-})$ does not appear in $(df)_{\underline{\lambda}}(J^{\underline{k}})$, thus $(df)_{\underline{\lambda}}(J^{\underline{k}}) = 0$, which is the same as $\gamma_{\underline{\lambda}}(J^{\underline{k}})$ by inductive assumption. This proves (4.17). Thus by replacing γ by $\gamma - df$, we have the lemma. $\square$

Lemma 4.6: $\gamma_{\underline{\lambda}}(J^{\underline{n}}) = 0$ *if* $q = 3$.

Proof: When $q = 3$, by Lemmas 4.3-5, we are left to consider the cases $\underline{n} = (0, n_2, n_2)$ and $\underline{n} = (0, n_2, n_2 + 1)$ for $n_2 \geq 1$. First suppose $\underline{n} = (0, n_2, n_2)$. As in (4.14), we have

$$0 = (d\gamma)_{\underline{\lambda}[4]}(J^0, J^0, J^{n_2}, J^{n_2+1}) = (n_2+1)(-\lambda_1\gamma_{\lambda_2,\lambda_3,\lambda_1+\lambda_4}(J^{\underline{n}}) + \lambda_2\gamma_{\lambda_1,\lambda_3,\lambda_2+\lambda_4}(J^{\underline{n}})). \tag{4.19}$$

Setting $\lambda_4 = 0$, it gives that $\gamma_{\underline{\lambda}}(J^{\underline{n}})$ can be divided by λ_1. So we can write $\gamma_{\underline{\lambda}}(J^{\underline{n}}) = \lambda_1\gamma'_{\underline{\lambda}}$ for some polynomial $\gamma'_{\underline{\lambda}}$, and (4.19) shows that $\gamma'_{\lambda_2,\lambda_3,\lambda_1+\lambda_4} = \gamma'_{\lambda_1,\lambda_3,\lambda_2+\lambda_4}$. Setting $\lambda_1 = 0$, this gives that $\gamma'_{\lambda_2,\lambda_3,\lambda_4} = \gamma'_{0,\lambda_3,\lambda_2+\lambda_4}$. Thus

$$\gamma_{\underline{\lambda}}(J^{\underline{n}}) = \lambda_1\gamma'_{0,\lambda_2,\lambda_1+\lambda_3}. \tag{4.20}$$

But $\gamma_{\underline{\lambda}}(J^{\underline{n}})$ is skew-symmetric with respect to λ_2, λ_3, we obtain $\gamma'_{0,\lambda_2,\lambda_1+\lambda_3} = -\gamma'_{0,\lambda_3,\lambda_1+\lambda_2}$. Setting $\lambda_1 = 0$ and $\lambda_3 = 0$ respectively, we obtain that $\gamma'_{0,\lambda_2,\lambda_3} = -\gamma'_{0,\lambda_3,\lambda_2}$ and $\gamma'_{0,\lambda_2,\lambda_1} = -\gamma'_{0,0,\lambda_1+\lambda_2}$, which gives that $\gamma'_{\underline{\lambda}} = 0$. Thus $\gamma_{\underline{\lambda}}(J^{\underline{n}}) = 0$.

Next suppose $\underline{n} = (0, n_2, n_2+1)$. We still have (4.20) for some polynomial $\gamma'_{\underline{\lambda}}$. We assume that $n_2 \geq 2$ (the proof for the case $n_2 = 1$ is similar and we leave it to the reader). For $1 \leq i < n_2$, by (2.13) and the inductive assumption, we have

$$\begin{aligned} 0 &= (d\gamma)_{\underline{\lambda}[4]}(J^0, J, J^{n_2-i}, J^{n_2+i+1}) \\ &= (n_2-i)\lambda_1\gamma_{\lambda_2,\lambda_1+\lambda_3,\lambda_4}(J, J^{n_2-i-1}, J^{n_2+i+1}) + (n_2+i+1)\lambda_1\gamma_{\lambda_2,\lambda_3,\lambda_1+\lambda_4}(J, J^{n_2-i}, J^{n_2+i}). \end{aligned} \tag{4.21}$$

Note that when $i = n_2-1$, the first term of the right-hand side is zero since $\gamma_{\underline{\lambda}}(J, J^0, J^{2n_2}) = -\gamma_{\lambda_2,\lambda_1,\lambda_3}(J^0, J, J^{2n_2})$ and $(0, 1, 2n_2) < \underline{n}$. Thus induction on i gives that $\gamma_{\underline{\lambda}}(J, J^{n_2-i}, J^{n_2+i}) = 0$. Then by (2.13) and the inductive assumption,

$$\begin{aligned} 0 &= (d\gamma)_{\underline{\lambda}[4]}(J^0, J, J^{n_2}, J^{n_2+1}) \\ &= -\lambda_1\gamma_{\lambda_1+\lambda_2,\lambda_3,\lambda_4}(J^{\underline{n}}) - (n_2+1)\lambda_1\gamma_{\lambda_2,\lambda_3,\lambda_1+\lambda_4}(J, J^{n_2}, J^{n_2}) \\ &\quad +(n_2\lambda_2 - \lambda_3)\gamma_{\lambda_1,\lambda_2+\lambda_3,\lambda_4}(J^{\underline{n}}) + ((n_2+1)\lambda_2 - \lambda_4)\gamma_{\lambda_1,\lambda_3,\lambda_2+\lambda_4}(J^{\underline{n}}). \end{aligned} \tag{4.22}$$

Substituting (4.20) into (4.22), cancelling the common factor λ_1, then setting $\lambda_1 = \lambda_2 = 0$, we obtain that $0 = -(n_2+1)\gamma_{0,\lambda_3,\lambda_4}(J, J^{n_2}, J^{n_2}) - (\lambda_3+\lambda_4)\gamma'_{0,\lambda_3,\lambda_4}$, which shows that $\gamma'_{0,\lambda_3,\lambda_4}$ is skew-symmetric with respect to λ_3, λ_4. Thus

$$f_{\lambda_1,\lambda_2}(J^{m_1}, J^{m_2}) = \begin{cases} -n_2^{-1}\gamma'_{0,\lambda_1,\lambda_2} & \text{if } (m_1, m_2) = (n_2, n_2), \\ 0 & \text{otherwise}, \end{cases} \tag{4.23}$$

defines a 2-cochain f (cf. (4.16)). Now as in the proof of Lemma 4.5, by replacing γ by $\gamma - df$, we have the lemma. This also proves (2.16). □

Lemma 4.7: (2.17) holds.

Proof: By Lemma 3.5, it remains to consider the case $q = 3$. Let $\bar{\gamma}$ be the 3-cocycle defined in (4.7). Let $\bar{\gamma}' = \Delta\bar{\gamma}$ be the corresponding reduced 3-cocycle (cf. (3.16)). Clearly $\bar{\gamma}'$ is nontrivial. Now suppose γ' is arbitrary reduced 3-cocycle. As in the paragraph before Lemma 4.3, we shall prove by induction on $\underline{n} \in \mathcal{N}_3$ that by replacing γ' by $\gamma' - c\bar{\gamma}' - df'$ for some $c \in \mathbb{C}$ and some reduced 2-cochain f' we have $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{m}}) = 0$ for $\underline{m} \leq \underline{n}$. Assume that we have proved $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{m}}) = 0$ for $\underline{m} < \underline{n}$.

First suppose $\underline{n} = (0, 0, n_3)$. By (2.13), (3.16) and the inductive assumption, we have

$$0 = (d\gamma')_{\underline{\lambda}[3]}(J^0, J^0, J^0, J^{n_3+1}) = (n_3+1)(-\lambda_1\gamma'_{\lambda_2,\lambda_3}(J^{\underline{n}}) + \lambda_2\gamma'_{\lambda_1,\lambda_3}(J^{\underline{n}}) - \lambda_3\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})). \tag{4.24}$$

Thus

$$\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}}) = \lambda_1\gamma'_{1,\lambda_2}(J^{\underline{n}}) - \lambda_2\gamma'_{1,\lambda_1}(J^{\underline{n}}). \tag{4.25}$$

If $n_3 = 0$, then by (2.12) and (3.16), $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}}) = -\gamma'_{\lambda_1,-\lambda_1-\lambda_2}(J^{\underline{n}})$, this together with (4.25)

gives that $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=0$. If $n_3=1$, then by (2.13), (3.16) and the inductive assumption,

$$\begin{aligned}0&=(d\gamma')_{\underline{\lambda}[3]}(J^0,J^0,J,J)\\&=-\lambda_1(\gamma'_{\lambda_2,\lambda_1+\lambda_3}(J^{\underline{n}})-\gamma'_{\lambda_2,-\lambda_2-\lambda_3}(J^{\underline{n}}))+\lambda_2(\gamma'_{\lambda_1,\lambda_2+\lambda_3}(J^{\underline{n}})-\gamma'_{\lambda_1,-\lambda_1-\lambda_3}(J^{\underline{n}})).\end{aligned}\tag{4.26}$$

Using (4.25) in (4.26), we see that γ'_{1,λ_1} can be divided by λ_1. Writing $\gamma'_{1,\lambda_1}=\lambda_1 p(\lambda_1)$ for some polynomial $p(\lambda_1)$ and using this in (4.26), cancelling the common factor $\lambda_1\lambda_2$, and setting $\lambda_2=\lambda_3=0$, we see that $p(\lambda_1)=c\in\mathbb{C}$ is a constant. Thus $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=c(\lambda_1-\lambda_2)$. Replacing γ' by $\gamma'+c\bar{\gamma}'$, we obtain that $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{m}})=0$ for $\underline{m}\le\underline{n}$.

If $n_3\ge 2$, we defines a reduced 2-cochain f' as follows:

$$f'_{\lambda_1}(J^{m_1},J^{m_2})=\begin{cases}-\gamma'_{1,\lambda_1}(J^{\underline{n}}) & \text{if } (m_1,m_2)=(0,n_3-1),\\ 0 & \text{otherwise,}\end{cases}\tag{4.27}$$

for $(m_1,m_2)\in\mathcal{N}_2$. Clearly, this indeed defines a reduced 2-cochain f'. Using (4.25), by replacing γ' by $\gamma'-df'$ as in the proof of Lemma 4.5, we have $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{m}})=0$ for $\underline{m}\le\underline{n}$.

Next suppose $\underline{n}=(0,n_2,n_2)$ for $n_2\ge1$. As in (4.25), from $(d\gamma')_{\underline{\lambda}[3]}(J^0,J^0,J^{n_2},J^{n_2+1})=0$ we obtain that $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=\lambda_1\gamma'_{1,\lambda_2}(J^{\underline{n}})$. But $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=-\gamma'_{\lambda_1,-\lambda_1-\lambda_2}(J^{\underline{n}})$ by (2.12) and (3.16), we obtain $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=0$.

Now suppose $\underline{n}=(0,n_2,n_2+1)$ for $n_2\ge1$. From $\gamma'_{\underline{\lambda}[3]}(J^{\underline{k}})=0$ for $\underline{k}=(0,0,n_2,n_2+2)$ and $\underline{k}=(0,0,n_2+1,n_2+1)$, we obtain that $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=\lambda_1\gamma'_{1,\lambda_2}(J^{\underline{n}})$ and that (4.26) again holds. From this, we obtain that $\gamma'_{\lambda_1,\lambda_2}(J^{\underline{n}})=-\gamma'_{\lambda_1,-\lambda_2}(J^{\underline{n}})$. Thus we can define a reduced 2-cochain f' such that $f'_{\lambda_1}(J^{m_1},J^{m_2})=\gamma'_{1,\lambda_1}(J^{\underline{n}})$ if $(m_1,m_2)=(n_2,n_2)$ or $f'_{\lambda_1}(J^{m_1},J^{m_2})=0$ otherwise. Then the rest of the proof is as before.

Finally suppose $\underline{n}=(0,n_2,n_3)$ with $n_3\ge n_2+2$ or $\underline{n}=(n_1,n_2,n_3)$ with $n_1\ge1$. Then the proof is the same as that of Lemmas 4.4 and 4.5. □

This completes the proof of Theorem 2.5.

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[1] C. Boyallian, V. G. Kac and J. I. Liberati, “On the classification of subalgebras of Cend_N and gc_N,” J. Algebra **260**, 32–63 (2003).

[2] B. Bakalov, V. G. Kac and A. A. Voronov, “Cohomology of conformal algebras,” Commun. Math. Phys. **200**, 561–598 (1999).

[3] S. J. Cheng and V. G. Kac, “Conformal Modules,” Asian J. Math. **1**, 181–193 (1997). Erratum, Asian J. Math. **2**, 153–156 (1998).

[4] S. J. Cheng, V. G. Kac and M. Wakimoto, "Extensions of conformal modules," in *Topological field theory, primitive forms and related topics,* Proceedings of Taniguchi and RIMS symposia, Progress in Math., Birkhäuser, 1998.

[5] A. D'Andrea and V. G. Kac, "Structure theory of finite conformal algebras," Selecta Math. **4**, 377–418 (1998).

[6] A. De Sole and V. G. Kac, "Subalgebras of gc_N and Jacobi polynomials," Canad. Math. Bull. **45**, 567–605 (2002).

[7] B. L. Feigin, "On the cohomology of the Lie algebra of vector fields and of the current algebra," Selecta Math. Soviet. **7**, 49–62 (1988).

[8] B. L. Feigin and D. B. Fuchs, "Homology of the Lie algebra of vector fields on the line," (Russian) Funkc. Anal. i Pril. **14**, 45–60 (1980).

[9] D. B. Fuchs, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986.

[10] D. Fattori and V. G. Kac, "Classification of finite simple Lie conformal superalgebras," preprint, math.QA/0106002

[11] I. M. Gelfand and D. B. Fuchs, "Cohomologies of the Lie algebra of formal vector fields," (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **34**, 322–337 (1970).

[12] V. G. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, Providence, RI, 1996.

[13] V. G. Kac, "The idea of locality," in *Physical applications and mathematical aspects of geometry, groups and algebras,* H.-D. Doebner et al, eds., World Sci., Singapore, 16–32 (1997).

[14] V. G. Kac, "Formal distribution algebras and conformal algebras," a talk at the Brisbane, in *Proc. XIIth International Congress of Mathematical Physics (ICMP '97) (Brisbane),* 80–97.

[15] B. Kostant, "Lie algebra cohomology and the generalized Borel-Weil theorem," Annals of Math. **74**, 329–387 (1961).

[16] W. Li, "2-Cocycles on the algebra of differential operators," J. Algebra **122**, 64–80 (1989).

[17] J. L. Loday, *Cyclic homology,* Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1992.

[18] Y. Su, "2-Cocycles on the Lie algebras of generalized differential operators," Commun. Alg. **30**, 763–782 (2002).

[19] Y. Su and K. Zhao, "Second cohomology group of generalized Cartan type W Lie algebras and central extensions," Commun. Alg. **30**, 3285–3309 (2002).

[20] X. Xu, "Equivalence of conformal superalgebras to Hamiltonian superoperators," Algebra Colloquium **8**, 63–92 (2001).

[21] X. Xu, "Simple conformal algebras generated by Jordan algebras," preprint, math.QA/0008224.

[22] X. Xu, "Simple conformal superalgebras of finite growth," Algebra Colloquium **7**, 205–240 (2000).

[23] X. Xu, Quadratic Conformal Superalgebras, *J. Algebra* **231**, 1–38 (2000).